to all the planes and all the directions in them, i.e., within the limits

$$0 \leqslant \alpha \leqslant 2\pi \qquad -\pi/2 \leqslant \varphi \leqslant \pi/2$$
$$-\pi/2 \leqslant \beta \leqslant \pi/2 \tag{9}$$

where

$$Q_1 = 1/4\pi \qquad Q_2 = 1/4\pi^2 \tag{10}$$

On the basis of this and relations (8), Eqs. (4) assume the form

$$\sigma_{ij} = \frac{a}{4\pi}\int_0^{2\pi} d\alpha \int_{-\pi/2}^{\pi/2} d\varphi[\epsilon_{nn}l_{in}l_{jn}] + \frac{b}{4\pi^2}\int_0^{2\pi} d\alpha \times$$
$$\int_{-\pi/2}^{\pi/2} d\varphi \int_{-\pi/2}^{\pi/2} d\beta[\epsilon_{nm}(l_{in}l_{jm} + l_{im}l_{jn})] \tag{11}$$

Taking into account Eqs. (6), after integration we get

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\epsilon_{ij} \tag{12}$$

that is, the ordinary generalization of Hooke's law, where

$$\lambda = (a + b)/15 \qquad \mu = (2a + 3b)/30 \tag{13}$$

On the basis of known equations connecting Lamé's constants with Young's modulus E and Poisson's ratio ν we get

$$E = \frac{a}{3}\left[\frac{2a + 3b}{4a + b}\right] \qquad \nu = \frac{a - b}{4a + b} \tag{14}$$

Whence it follows that

$$a = 3E/(1 - 2\nu) = 3K \tag{15}$$

Here K is the modulus of bulk compression.

Suppose that the yield stress, measured in a uniaxial tension test, is σ_H, whereas the theoretical stress, calculated from conditions of rupture of the atomic planes, is σ_T. The maximum tearing stresses are experienced by atomic planes perpendicular to the direction in which the tension is applied. By virtue of relations (6)

$$\sigma_{nn} = \sigma_T = a\epsilon_{nn} = a\epsilon \tag{16}$$

where ϵ is the axial deformation. On the other hand, $\epsilon = \sigma_0/E;$ but then on the basis of Eq. (16) we have

$$\sigma_H = 3(1 - 2\nu)\sigma_T \tag{17}$$

From this it follows that, as ν approaches $\frac{1}{2}$ (incompressible body), the observed rupture stress may be much less than the theoretical stress. Although (in view of the asymmetry of the method with respect to stresses and deformations) this cannot be regarded as a new explanation of the sharp difference between the theoretical and observed rupture stresses, a more critical approach is necessary to existing calculations of theoretical stress with regard to interactions not coinciding with the direction of rupture.

—*Submitted May 18, 1961*

JULY 1963 AIAA JOURNAL VOL. 1, NO. 7

Cycles about a Singular Point of Nodal Type

A. P. Vorob'ev
Institute for Mathematics and Computational Technology, Academy of Sciences, White Russian Soviet Republic

***In a differential equation system in the plane**

$$dx/dt = P(x, y) \qquad dy/dt = Q(x, y)$$

where P and Q are polynomials, with $P(0,0) = Q(0,0) = 0$, there may exist closed trajectories enclosing the singular point at the origin. If the degrees of P and Q are at most two, and the origin is a nodal point, it is shown that there can be no closed trajectory enclosing the origin. Sufficient conditions for a nodal point to be acyclic (i.e., not to be enclosed by a closed trajectory) are developed.

It is known that for a system of differential equations of the form

$$\frac{dx}{dt} = \sum_{i+j=1}^{n} a_{ij}x^iy^j$$
$$\frac{dy}{dt} = \sum_{i+j=1}^{n} b_{ij}x^iy^j \tag{1}$$

(a_{ij},b_{ij} constant) that when $n = 1$ (a linear system) a singular point of nodal type cannot be enclosed by a closed trajectory. This is possible for $n \geqq 3$. For example, in the system

$$dx/dt = -y + ax(x^2 + y^2 - 1)$$
$$dy/dt = x + by(x^2 + y^2 - 1)$$

with $ab \geqq -1$, $(a - b)^2 \geqq 4$, the singular point at (0,0) is a node, and it is enclosed by the closed trajectory $x^2 + y^2 = 1$.

In the present note we shall formulate sufficient conditions that a singular point be acyclic,† and shall show that in the system (1) with $n = 2$ a singular point of nodal type cannot be enclosed by a closed solution.

Consider the system of differential equations

$$d\rho/dt = F(\rho,\theta) \qquad d\theta/dt = \Phi(\rho,\theta) \tag{2}$$

(where ρ and θ are polar coordinates and $F(0,\theta) \equiv 0$), for which conditions for existence and uniqueness of solutions are fulfilled throughout the entire plane.

Translated from Doklady Akademii Nauk BSSR (Proceedings of the Academy of Sciences, White Russian Soviet Republic) 4, No. 9, 369–371 (1960). Translated by Warren S. Loud, Department of Mathematics, University of Minnesota, Minneapolis, Minn. Reviewed for publication by Lawrence Markus, Department of Mathematics, University of Minnesota, Minneapolis, Minn.

* This abstract was prepared by Professor Warren S. Loud.

† By an acyclic singular point we shall mean a singular point not enclosed by a closed trajectory.

Lemma: Let the graph of the equation $\Phi(\rho,\theta) = 0$ have a branch Γ, a portion of which, Γ_1, extends to infinity. Let Γ_1 be contained in the sector $\theta_1 \leqq \theta \leqq \theta_2$ where $\theta_2 - \theta_1 < 2\pi$. Let $\Phi(\rho,\theta)$ change sign as Γ_1 is crossed, and let there be no other branch in this sector having this property. Then no closed trajectory which encloses the origin can have points in common with Γ_1.

Proof: At points of Γ_1 where $F(\rho,\theta) \neq 0$, the direction field of integral curves is tangent to the radius vector drawn from the origin. Since $\Phi(\rho,\theta)$ changes sign as Γ_1 is crossed, the coordinate θ on the integral curves has an extreme value when the integral curve meets Γ_1. Also, since there is no other branch of $\Phi(\rho,\theta) = 0$ in the sector, on crossing which $\Phi(\rho,\theta)$ changes sign, a trajectory entering this sector will always remain within it. This proves the lemma.

Corollary: If the graph of the equation $\Phi(\rho,\theta) = 0$ has a unique branch in the sector $\theta_1 \leqq \theta \leqq \theta_2$, with one end at the origin and the other extending to infinity, and if $\Phi(\rho,\theta)$ changes sign on crossing this branch, then the origin is acyclic.

Let us now consider the system of differential equations

$$\begin{aligned} dx/dt &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ dy/dt &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{aligned} \tag{3}$$

with constant coefficients. The following theorem holds.

Theorem: In the system (3) a singular point of nodal type cannot be enclosed by a closed trajectory.

Proof: Let us assume that the origin is a singular point of nodal type, and let us introduce a system of polar coordinates. The transformed system has the form

$$\begin{aligned} d\rho/dt = \rho\{&a_{10}\cos^2\theta + (a_{01} + b_{10}) \cos\theta \sin\theta + \\ &b_{01}\sin^2\theta + \rho[a_{20}\cos^3\theta + (a_{11} + b_{20})\cos^2\theta \sin\theta + \\ &(a_{02} + b_{11})\cos\theta \sin^2\theta + b_{02} \sin^3\theta]\} \end{aligned} \tag{4}$$

$$\begin{aligned} d\theta/dt = &b_{10} \cos^2\theta + (b_{01} - a_{10}) \cos\theta \sin\theta - \\ &a_{01} \sin^2\theta + \rho[b_{20}\cos^3\theta + (b_{11} - a_{20}) \times \\ &\cos^2\theta \sin\theta + (b_{02} - a_{11}) \cos\theta \sin^2\theta - \\ &a_{02} \sin^3\theta] \equiv f_2(\theta) + \rho f_3(\theta) \equiv \Phi(\rho,\theta) \end{aligned}$$

Since $x = 0$, $y = 0$ is a singular point of nodal type, $f_2(\theta)$ either vanishes identically or has a real zero, θ_1. The equation $f_3(\theta) = 0$, since it is a homogeneous polynomial of third degree in $\sin\theta$ and $\cos\theta$, always has at least one real root. If $f_2(\theta) \equiv 0$, if $f_3(\theta) \equiv 0$, or if $f_2(\theta_1) = 0$ and $f_3(\theta_1) = 0$, than the conclusion of the theorem follows from the fact that the system (3) has at least one integral curve which is a straight line through the origin. It thus remains to consider the case when the equations $f_2(\theta) = 0$ and $f_3(\theta) = 0$ have no common root. Clearly we can always find a sector (θ_1,θ_2) such that $f_2(\theta_1) = 0$ and $f_3(\theta_2) = 0$, and such that neither $f_2(\theta)$ nor $f_3(\theta)$ vanishes for any other value of θ in this sector. Consider the curve $\rho = -f_2(\theta)/f_3(\theta)$ in this sector. Without loss of generality we may assume that the function $-f_2(\theta)/f_3(\theta)$ is positive within the sector, since if it were negative, we would consider the opposite sector $(\theta_1 \pm \pi, \theta_2 \pm \pi)$, in which it would be positive. Clearly $\rho = -f_2(\theta)/f_3(\theta)$ represents a branch of the graph of $\Phi(\rho,\theta) = 0$ for the system (4), which has one end at the origin and the other extending to infinity, wholly contained in a sector less than 2π. As for the variation of sign of $\Phi(\rho,\theta)$ on crossing this branch, this follows from the linearity of $\Phi(\rho,\theta)$ in ρ. There are no other branches of $\Phi(\rho,\theta) = 0$ in this sector. Thus the conditions of the corollary to the lemma are satisfied. The theorem is proved.

Taking into account the fact that the system (3) can have no more than two singular points of "focus" type,[1] we obtain, as a corollary to the theorem just proved, a known result:[2] If the system (3) has three limit cycles, $L_c^i (i = 1,2,3)$, then it is impossible for $I_c^i \cap I_c^j = 0$ for $i \neq j,i,j = 1,2,3$. (I_c^k denotes the domain enclosed by L_c^k.)

We mention in conclusion that this problem was proposed in the seminar of N. P. Erugin and Yu. S. Bogdenov. The forementioned example was presented there.

—*Submitted March 1, 1960*

References

[1] Berlinskii, A. N., Memoirs of the Stalingrad State Pedagogical Institute **XX**, 3 (1958).

[2] Chin-Chu, Tung, Acta Mathematatica Sinica **8**, no. 2 (1958).

JULY 1963 AIAA JOURNAL VOL. 1, NO. 7

Convergence of a Generalized Interpolation Polynomial

V. N. Rusak

In his paper[1] S. N. Bernshtein proved that the generalized polynomial which deviates least from zero on the interval $[-1,1]$ among all generalized polynomials of the form

$$\frac{x^n + b_1x^{n-1} + \ldots + b_n}{\prod_{i=1}^{n}\left(1 - \dfrac{x}{a_i}\right)} = A_0 + \sum_{k=1}^{n} \frac{A_k}{a_k - x}$$

has the representation

$$r_n^*(x) = C \cos(n\theta + \psi) \tag{1}$$

where C is a definite constant

$$\cos\theta = x \qquad \psi = 2 \sum_{k=1}^{n} \alpha_k$$

$$\cos\alpha_k = \frac{\rho_k - x}{\sqrt{2\rho_k(a_k - x)}}$$

$$\sin\alpha_k = \frac{\sqrt{1 - x^2}}{\sqrt{2\rho_k(a_k - x)}}$$

$$a_k = \frac{1}{2}\left(\rho_k + \frac{1}{\rho_k}\right)$$

$$|\rho_k| > 1 \qquad k = 1, 2, \ldots, n$$

We consider generalized interpolation polynomials on the interval $[-1,1]$ with respect to the system of Chebyshev functions

$$\frac{1}{a_1 - x}, \quad \frac{1}{a_2 - x}, \ldots, \quad \frac{1}{a_n - x} \tag{2}$$

$$|a_i| > 1 \qquad i = 1, 2, \ldots, n$$

Translated from Doklady Akademii Nauk Belorussia SSR (Reports of the Academy of Sciences of the Belorussian SSR), VI, no. 4, 209–211 (1962). Translated by Warren S. Loud. Professor of Mathematics, University of Minnesota.